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OPTIMAL GROWTH IN A TWO-SECTOR RSS MODEL WITHOUT DISCOUNTING: A GEOMETRIC INVESTIGATION*

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This essay reports results on optimal growth in a two-sector model with fixed coefficients, irreversible investment and no discounting. Under normalization, the model can be represented by two real numbers, but despite its deceptive simplicity, it admits rich transition dynamics and apparent pathologies that seem to have been missed in earlier work. From a methodological point of view, and in the light of recent work of Nishimura and Yano, this essay can also be seen as a further rehabilitation of geometric methods as an engine of analysis.
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1. Introduction

In a recent investigation of the "choice of technique" in a vintage-capital model originally formulated by Robinson, Solow and Srinivasan (henceforth, the RSS model), and further discussed by Okishio and Stiglitz,[1] it was shown that a single technique played an increasingly dominating and singular role in the long run, despite the availability of a multiplicity of techniques; see Khan and Mitra (2003). In a world without discounting and under a standing hypothesis on technological parameters, a single type of machine, one independent of the parameters of the felicity function, was produced and used in the long run; even if a (well identified) class of types of machines were available, they were never replaced but used until their stocks depreciated to zero; see (Khan and Mitra (2003), sections 3–5). However, it was also shown that these results do not extend to the medium and the shortrun, and even in an economy with two types of machines, with a single type being optimal in the long run, both types were optimally produced and used in the first period. Whereas sufficient conditions on the parameters of the technology were furnished to rule out this

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[1] In addition to the pioneering analyses of Robinson (1960, pp. 38–56), Solow (2000) and Srinivasan (1962), see Okishio (1966), Bruno (1967), Stiglitz (1968, 1970, 1973b), Robinson (1969), Cass-Stiglitz (1970) and Solow (2000). In Khan and Mitra (2003), an attempt is made to give some perspective to this rich literature.

phenomenon, the complete characterization of the transition dynamics was left open; see (Khan and Mitra (2003), Section 8).

In this paper, we present a complete resolution of this problem for an economy with only one type of machine and with a linear felicity function, remaining in a world with discrete time and with zero discounting. In such an economy, there is no question as to the choice of technique, and it is natural to ask whether, with all the simplifications—the model's parameters are reduced to two real postive numbers—there is anything very much left to determine; the problem seems at first glance to have been simplified into a triviality. However, even in this starkly simplified special case of the RSS model, and even of the standard two-sector model, the results go against established intuitions in a way that is nothing short of dramatic. A parameter ξ that seemed to play a peripheral role in the counterexamples presented in (Khan and Mitra (2003), subsections 6.2 and 6.3) now attains a central role in the analysis; what was seen as a possibility of periodic optimal programmes and of programmes giving priority to the consumption sector as being "bad" in the sense of generating infinite value-losses,[2] now emerge as robust results under specific ranges of this parameter.

In the economy under consideration, a units of perfectly divisible labour are required to produce a perfectly divisible machine with a depreciation rate of d. What is crucially important for the transition dynamics is the maximal amount of incremental machines that is available tomorrow with a unit stock of machines today (ξ equal to $1/a - (1 - d)$).[3] This constant parameter ξ can only take values greater than -1,[4] and when it is unity, an optimal programme consists of two-period cycles around a unique golden rule stock of machines, with the amplitude of each cycle being different for different values of the initial stock of machines lying in an identifiable interval. Outside this interval, there is convergence in finite time to one of these cycles, but a convergence that is typically not monotonic. However, what is especially interesting about this case is that the optimal programme is not unique; there exists an interval of initial stocks in which the optimal policy is constituted by a set of values rather than a singleton. This leads to an optimal policy correspondence rather than to an optimal policy function.

For other values of ξ, we can show the existence of an optimal policy function; in each period, there is a unique value of the optimal amount of machines, and therefore of an optimal level of consumption and of optimal (sectoral) labour allocation, as a function of the amount of machines available in the previous period. We show that the optimal programme converges to the golden rule stock, and in particular, for values of ξ greater than unity, and for the initial stocks of machines lying in an interval between the golden rule stock and unity,[5] there is convergence in the very first period. For other values of ξ, however, convergence is again not necessarily of a monotonic type, and takes finite or infinite time depending on the particular values of the initial stock of machines the economy is endowed with. Thus, in several instances, Ramsey optimality requires over-building and under-depreciating relative to the golden rule stock. In particular, the dynamic system fails in a particularly sharp way the recent criterion of "history independence" proposed

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[2] Such programmes are advocated by Stiglitz as a consequence of his analysis in continuous time; see Stiglitz (1968) and also Khan and Mitra (2003).

[3] As we shall see below, the exogenous (and identical) amount of labour available to the economy in each period is normalized to unity.

[4] If $\xi \leq -1$, $d \leq -(1/a)$, a contradiction to the positivity of a, given that $0 < d < 1$.

[5] As emphasized in footnote 3, labour has been normalized to be unity in each period.

in Mitra and Nishimura (2001b). In any case, we give a complete (global) characterization of the optimal programme.

It is worth noting that even though the results appear surprising in terms of the volatility of the optimal programmes when viewed against the backdrop of earlier work that emphasized monotonic convergence,[6] they seem tame in the light of more recent work on chaotic dynamics two-sector models, both of the standard Srinivasan-Uzawa type (see the papers collected in Benhabib (1992) and Majumdar *et al.* (2000)), and those incorporating externalities and endogenous growth (see the JET Symposium introduced in Mitra and Nishimura (2001a)). Our model, despite its simplicity, is a bona fide two-sector model; robust methods are available in this literature which allow one to construct two-sector models whose optimal programmes exhibit chaos under its various definitions. It is these programmes that really merit the term "volatile" and the results presented here can be read as emphasizing the lack of volatility of the optimal programmes when there is no discounting and one of the sectors, the investment sector, uses only one factor, namely labour.[7] In this sense, our results are in keeping with an early intuition of Lionel McKenzie's that asymptotic theory for optimal paths of capital accumulation is "rather fully developed" for the undiscounted case, and more difficult for the discounted case; see McKenzie (1983).

Whereas the results presented in this paper obviously constitute its primary motivation, an important secondary motivation is a rehabilitation of geometric methods—we give them prominence as an engine of analysis rather than the somewhat subsidiary role that they have played in the bulk of the literature as an instrument of illustration. We rely on the basics of Euclidean geometry, as exposited for example in Herberg and Orleans (1940) or Artmann (1999), and as such use techniques well-known to the ancient Greeks. In keeping with Khan and Mitra (2003), we work with the Gale-McKenzie reduced form, and combine a cobweb-like analysis with the period by period diagram especially prominent in McKenzie's work.[8] In this move beyond calculus to a recourse to global methods, we are very much in the tradition of the classical theory of international trade,[9] and also in tune with a similar move in economic dynamics pioneered by Nishimura and Yano. We defer a detailed discussion of the Nishimura-Yano geometry to the sequel, and merely observe here that our methods play an important initial role in allowing us to go beyond general statements concerning transition dynamics to a detailed (global) characterization of the structure of the optimal policy. We can compute the number of periods it takes for an optimal policy to converge to the golden rule stock, and explicitly graph their dependence

6 See, for example, Stiglitz (1968) in the context of a related model, and Koopmans (1967a, 1967b) for a general class of one-sector models.

7 However, it is worth observing that in either case, the full implications of all of this work have yet to be drawn out for the field which originally served as their primary motivation—development economics.

8 However, here too the diagrammatic methods are primarily illustrative, as in, for example, McKenzie (1999, Fig. 7). However, McKenzie (1983, Fig. 1) may possibly be an exception, but it is a setting with discounting. We leave it for future work to show how Liviatan's geometry for the one-sector model in Liviatan (1970) can be supplemented by one pertaining to the reduced form.

9 Such a tradition of exposition and investigation of course begins at least with Marshall's "Pure Theory of Foreign Trade" and continues with Samuelson, Meade, Johnson, Bhagwati, Jones, Findlay and Corden among others. In the context of growth theory, see Koopmans (1964). For a succinct statement on the advantages of the geometric method, see for example Marshall (1879, p. 5) and Johnson (1971, pp. 9–10). For differing reactions to the use of geometry, see Kurz's (1970) and Shackle's (1956) reviews of Meade's work and Koopmans (1957, essay 3) and Koopmans (1964). We owe the first Koopmans reference to Ron Jones.

on the range of the initial stocks of capital. In particular, even in situations where, in general, there is convergence to the golden rule stock in an infinite number of periods, we can identify a set of initial stocks, admittedly negligible, in the sense of a zero Lebesgue measure. Furthermore, by showing how basic and beautiful ideas in the general theory of intertemporal resource allocation are given a concrete form amenable to geometric manipulation, albeit in the specific context of our simple model, we hope to make them accessible to a wider audience.[10]

We conclude this introduction with a schematic outline of the paper. In Section 2, we develop the model and place it in the context of the two-sector neoclassical setting originally associated with Srinivasan and Uzawa.[11] Sections 3 and 4 lay out the basic geometrical apparatus that is put to use in Sections 5 to 9 to chart out the optimal policies under the various values of the parameter ξ. Section 10 relates our geometry to that of Mitra and Nishimura and concludes. Our geometrical analysis rests on two basic mathematical results: the Kuhn-Tucker theorem of optimization theory, and the Brock theorem on the existence of optimal programmes obtained as those minimizing aggregate value losses at suitably defined prices; see Uzawa (1958) and Brock (1970) respectively. For the reader's convenience, we present a (slightly extended) version of the latter in an Appendix.

2. The model and its antecedents

As already emphasized in the introduction, the special case of the RSS model that we analyse in this paper is also a special instance of the standard two-sector model of optimum economic growth.[12] There is a consumption and an investment sector, and at any moment in time, the planner allocates an exogenously given amount of capital and labour to either sector, and which, after taking an exogenously given rate of depreciation into account, yields a new level of capital in the next period. This, combined with an exogenously given growth rate of labour (assumed to be zero), allows a repetition of the process. Given the planner's objective function, and an initial stock of capital and labour, questions relating to an optimum programme can be posed and studied.

The model is distinguished by the fact that there is only a single technique of production in each of its two sectors, and that the investment sector uses only labour. Let $a > 0$ units of labour be required to produce a single machine and that a unit of labour can be used with a single machine to produce a unit of output. We shall assume both labour and machines (capital) to be perfectly divisible, and normalize labour availability to be unity in each period.[13]

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[10] For a quick list of the concepts exposited in this paper, the reader can see the last paragraph of the next section. Given the modern emphasis on recursive methods, none of these concepts are included, for example, to the textbook of Stokey and Lucas (1989), or the Majumdar *et al*. monograph (2000), for example.

[11] Also see Shell (1967) and Haque (1970) in the context of optimum growth. The use of the two-sector model in the theory of international trade has a long tradition; see Jones (1959b), Johnson (1971) and their references.

[12] In addition to Srinivasan (1964), Uzawa (1964), Shell (1967) and Haque (1970) in the case of optimum growth with discounting; also see Mitra (2000) for references to modern work. As is well known, the model has been used and exposited by Uzawa, Solow, Meade, Johnson and others in the context of so-called descriptive growth. We also give no references to the long tradition of the use of this model in Heckscher-Ohlin-Samuelson trade theory, as exposited for example in Jones (1965b) and Johnson (1971).

[13] The latter assumption seems to be in keeping with the tone of the time; see Aghion and Howitt (1988, Section 1.2) for the exposition of what is termed there as the Cass-Koopmans-Ramsey model. Such an assumption is also made by Srinivasan (1962) and by Stiglitz (1968, 1973a, 1973b).

With these simplifications, the amount of machines used in the consumption sector, $y(t)$, is identical to the amount of labour being used in that sector and to the amount of the consumption good $c(t + 1)$ being produced. Since we also assume that the felicity function $w(\cdot)$ to be linear and normalized so that $w(c(t + 1)) = c(t + 1)$, $y(t)$ also represents the utility levels obtained in the next period. If we let $x(t)$ be the amount of machines available in this period, and $z(t + 1)$ the amount of machines in production in this period (to be made available next period), we obtain

$$0 \leq y(t) \leq x(t) \text{ (the capital constraint) and; } 0 \leq y(t) \leq 1 - az(t + 1) \text{ (the labour constraint).} \tag{1}$$

Thus the total amount of machines available in the next period is given by

$$x(t + 1) = z(t + 1) + (1 - d)x(t) \Rightarrow y(t) \leq 1 - a(x(t + 1) - (1 - d)x(t)). \tag{2}$$

Thus, in the language of control theory, $x(t)$ represent the state variable, and $(c(t + 1), z(t + 1)) = (y(t), z(t + 1))$ the control variables, with the control set from which they are to be chosen, given by Equation (1). All that remains then is a precise specification of the planner's objective function.

In supposing that future welfare levels are treated similar to current ones in the planner's objective function, we take our lead from Ramsey (1928), but rather than the assumption of a "bliss point" or that of "capital saturation", as in Ramsey (1928), Samuelson and Solow (1956) and Samuelson (1965), we work with the "overtaking criterion of optimality" of Atsumi (1965) and von Weiszäcker (1965). Towards this end, given an initial capital stock $x_0 \geq 0$, we work with programs starting from x_0. These are simply non-negative sequences of capital stocks $\{y(t), x(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$ and which satisfy the technological and material balance constraints laid out in Equations (1) and (2) above. A programme $\{y^*(t), x^*(t)\}_{t=0}^{\infty}$ starting from x_0 is said to be *optimal* if there does not exist any other programme $\{y(t), x(t)\}_{t=0}^{\infty}$ starting from x_0 that overtakes it, which is to say that there does not exist any $\varepsilon > 0$, and a time period t_ε such that

$$\sum_{t=0}^{T} (y(t) - y^*(t)) \geq \varepsilon \quad \text{for all} \quad T \geq t_\varepsilon.$$

Thus an optimal programme is one in comparison to which no other programme from the same initial stock is eventually significantly better, for any given level of significance. A programme[14] $\{y(t), x(t)\}_{t=0}^{\infty}$ is said to be stationary if it is constant over time, i.e., $(y(t + 1), x(t + 1)) = (y(t), x(t))$ for all $t = 0, 1, \ldots$. A programme is said to be a stationary optimal programme if it is stationary and optimal.

Through our emphasis on our model as a special instance of the neoclassical model of optimal growth, we under-emphasize the assumption of fixed coefficient technology. It is this assumption that takes its lead from Von Neumann (1935–36) and looks towards the multi-sectoral setting in the work of Gale (1967a), Brock (1970) and McKenzie (1968), finding its culmination in the so-called "reduced form" model.[15] This is summarized by

[14] If we do not specify the starting point of a programme, the reader should take it to mean that it starts from its value at time zero.

[15] McKenzie (1999, p. 389) dates this "reduced form" model to 1964, and describes the work of Gale, McFadden and himself as a "fusion of the Ramsey and von Neumann models". Malinvaud's 1953 paper is also an important step in this evolution. For a comprehensive survey of the discounted setting, see Mitra (2000).

two basic parameters: a period to period technology set in a (product) space of capital stocks initial and terminal to the period, and a planner's utility function defined on this set.[16] Such a formulation is flexible enough to yield as special cases the variety of growth models studied in the literature and thereby qualify as a general theory of intertemporal resource allocation,[17] and it is through the vocabulary of this theory that we analyse the simple two-sector model studied here.[18] In addition to Brock's theorem, such a formulation leads us to discuss the golden rule capital stock and its associated golden rule price system, value loss per period and its aggregate over time, the von Neumann facet and its privileged subset termed here the McKenzie facet,[19] and the turnpike and average turnpike properties. In terms of a comparison with the methods of Pontryagin, as used in our context in Stiglitz (1968), perhaps the crucial difference is the possibility of (a complete) analysis based on the golden rule price system from the very start, rather a price system that corresponds to the optimal programme, and is then shown to converge to it.[20]

3. The basics of the geometry

We begin with Figure 1 in which the 45° line serves as an important benchmark. The number of machines, $x(t)$, available today is measured on the X-axis and that available next period, say tomorrow, $x(t + 1)$, on the Y-axis. Figure 1 highlights the fact that a machine is perfectly divisible[21] and that we work in discrete time. When the particular time period is not of any consequence, we shall use the symbols (x, x') for $(x(t), x(t + 1))$.

The line *OD*, lying strictly within the cone formed by the X-axis and the 45° line, represents a constraint embodying a precise time invariant form of depreciation, one that does not distinguish between machines produced at different points in time, as well as the fact that they cannot be disposed off through a market or in any other way. It alerts the reader to the fact that we work with irreversible investment and that there is a bound to disinvestment. The slope of *OD* measures the rate of depreciation;[22] a unit of today's capital stock depreciates to $(1 - d)$ units tomorrow.

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16 In his work, McKenzie does not give a name for the analogue in a general setting of the set that we denote by Ω. In Nishimura and Yano (1995), it is referred to as the transition possibility set. The evolution of this set is of some interest: in McKenzie (1968), it includes values of utilities, while McKenzie (1971) retains the notational convention of Debreu (1961) where positive numbers stand for outputs and negative ones for inputs.

17 For statements of the theory and its scope, see McKenzie (1986), McKenzie (1987), McKenzie (1999) and Mitra (2000).

18 As we shall indicate in the sequel, the relevant theorems of this theory do not directly apply, but the methods leading to the proofs of these theorems extend to our simple setting in a straightforward way.

19 This terminology is being introduced here for the first time; see Khan and Mitra (2002) for a textual substantiation. Note also the somewhat more expansive definition of von Neumann facets in Nishimura and Yano (2000, p. 264; 1995, p. 994).

20 This is of course not to say that a price system associated with any optimal programme has no role to play in the theory—only that we make no recourse to it for our present purposes.

21 In the sequel, we shall refer to the number of machines interchangeably as the capital stock.

22 The equation of this line is simply $x' = (1 - d)x$.

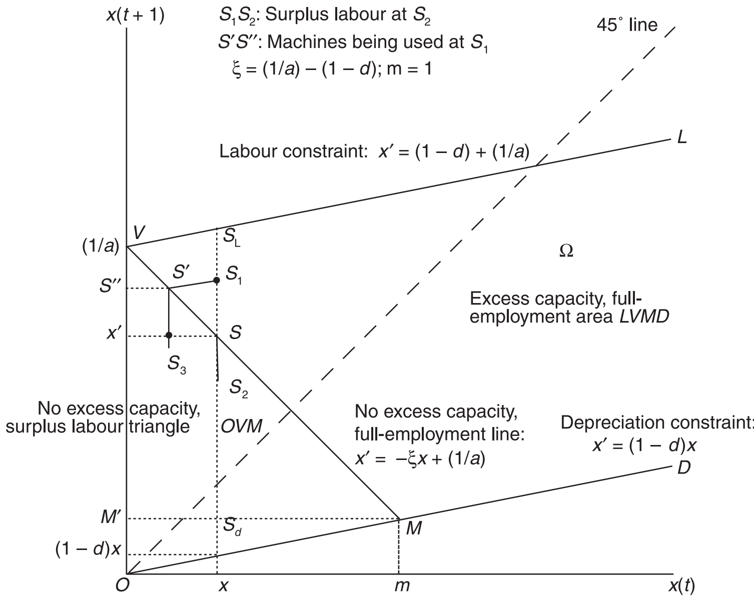

FIGURE 1. Specification of the period to period production set Ω

The line *VL* is obtained by adding the length[23] *OV* to the line *OD*. The segment *OV* equals $1/a$ and represents the maximal amount of capital stock available tomorrow if there is no capital stock available today and all of the labour force is employed. Since this maximal amount is a constant independent of the initial capital stock, *VL* is parallel to *OD*—in addition to the constraint on the availability of labour in each period, it reflects the fact that machines are not needed to manufacture machines.[24]

The "open" parallelogram Ω enclosed by the lines *OV*, *VL* and *OD* represents the basic technological specification of the model.[25] We shall refer to an element of this period to period production set Ω, also referred to as the transition set, as a plan.[26] Note that a plan implies a specification of feasible consumption levels, and as such, different from production plans in the sense of Debreu (1961). In this connection, two other features of Ω ought to be noted: (i) the absence of free disposal, already mentioned above; and (ii) a zero initial stock (input) allows a positive terminal stock (next period's output), specifically *OV*, next period.[27]

The line segment *MV* is a "most valuable" construct for the geometric development of the one-machine version of the RSS model. Note that to begin with, the absolute value of the slope of *MV* is precisely the parameter ξ singled out in the introduction,

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23 The reader is warned to be alert to identical symbolism being used to designate lengths and lines.

24 The equation of this line is simply $x' = (1 - d)x + 1/a$.

25 Formally, $\Omega = \{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1 - d)x \geq 0 \text{ and } a(x' - (1 - d)x) \leq 1\}$.

26 We shall not resort to the qualifying adjective "feasible" when we refer to a plan.

27 This does not of course imply a "free lunch"—labour which is kept in the background, is being used to produce the output. McKenzie's intuition of this set (1999, Figures 7 and 8), for example, bears comparison with Figure 1.

$$\frac{VM'}{MM'} = 1/a - (1 - d) \equiv \xi. \tag{3}$$

More generally, for any plan S on MV with generic coordinates (x, x'), we can appeal to properties of similar triangles,[28] to obtain the equation of the line MV,

$$\frac{x' - (1-d)x}{1/a} = \frac{SS_d}{OV} = \frac{MS_d}{OM} = 1 - x \Rightarrow 1 - a(x' - (1-d)x) = x \Rightarrow x' = (1/a) - \xi x. \tag{4}$$

It is important to note that the second form of the equation of MV equates the two constraints presented as Equations (1) and (2), and thus represents a situation where the amount of labour needed to attain the terminal capital stock x' leaves as a residual precisely the amount to fully utilize the initial capital stock x in the consumption sector. As such, MV is the locus of initial and final terminal stocks for which there is full employment and no excess capacity of capital, and hence it can be referred to as the *full employment*, *no excess-capacity* line.

Since full employment of capital and labour is a crucial consideration and cannot be automatically presumed in the model, a further elabouration of MV is warranted in this connection. Note that it divides Ω into two parts: the triangle MOV and the "open" remainder $LV\,MD$. Consider any plan in the triangle MOV not on MV, say S_2, and note that the amount of terminal capital stock required to generate full employment is given by the ordinate of the plan S; since net investment at S_2 is only S_2S_d, the ordinate of S_2 falls short by the amount SS_2. This segment SS_2 then is a measure of surplus labour. Similarly, any other plan of Ω "below" the line MV, the plans S_3 and S_d for example, all machines are being utilized in the consumption sector, but there is not enough activity in the investment sector to support full employment, and hence the presence of surplus labour (in the amount $S_1'S_3$ at S_3). At plans "above" MV, plans S_1 and S_L in $LV\,MD$ for example, the situation is reversed in the sense that there is not enough labour left over from the investment sector to utilize all of the available machines, and hence some machines are being left idle with resulting excess capacity. The question is how much of the available capital stock is then being used in the consumption sector.

The technological specification of the consumption sector, is not explicitly graphed in Figure 1,[29] but the constraints in Equations (1) and (2) allow us to express a fixed level of labour employment in the consumption sector, say y_2 as a line parallel to VL and MD and terminating at MV in a downward vertical, as exhibited in Figure 2.[30] This allows us to see the kinked lines OVL and mMD as zero and unit iso-employment lines in that sector, and MV in its second identity—the segment of the transition set Ω on which these iso-employment lines are pegged.

28 Our prerequisite is a most basic one; for a detailed treatment, the reader can see Herberg and Orleans (1940, Chapter IX) or Artmann (1999, Chapter 15).

29 McKenzie (1976, p. 846) writes "When our interest is an asymptotic property of the path of capital stocks, there is no need to show how utility depends on production and consumption during the period . . . the significant choice from the viewpoint of the intertemporal maximization problem is the choice of terminal stocks given initial stocks. This fixes the contribution of the period to the optimal programme." However, as we shall see below, in our specific context, the diagram can be useful even for the delineation of consumption levels and of dynamics in the short run.

30 In set-theoretic notation, the y_2 iso-employment line is given by $\{(x, x' \in \Omega: y_2 \le x \text{ and } x' = (1 - d)x + (1 - y_2)/a\}$.

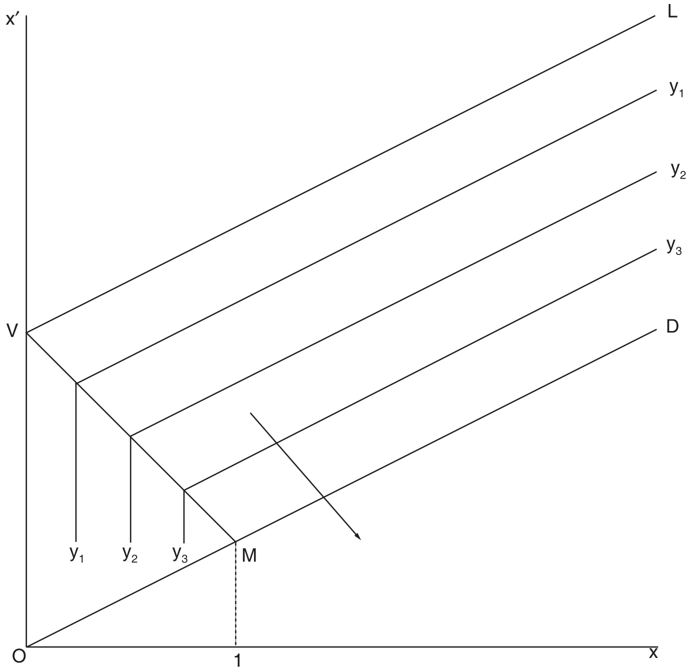

FIGURE 2. Iso-employment lines and the specification of planner's preferences

We can now use these iso-employment lines to compute the excess capacity for any plan in *LV MD*. Consider the plan S_1, and note that its iso-employment line is given (in part) by $S_3S_1'S$. At S_1 there is more investment activity that allows all the machines to be utilized, and only the amount of employment designated by the abscissa of S_1' is available for the consumption sector. Hence at S_1, there is full employment of labour, but idle capacity of the amount $x - S_1'S_1''$; only part of the stock of available capital, as measured by $S_1'S_1''$ is used. Similarly, at the one-period plan, S_L, zero capacity utilized is while at S it is unity with all the available capital stock being used.

It is worth noting at this stage that at a given initial capital stock x, the movement from S_L to S_d traces out a production possibility surface in the space of consumption and investment goods, as pictured in Figure 3. It is useful to see how the arguments furnished above in the context of Figure 1 translate to Figure 3. If all labour is employed in the investment sector, the economy is at S_L in Figure 3, with complete specialization, full employment and zero capital utilization. As we move down the production possibility surface, decreased investment and the consequent release of labour leads to the production of the consumption good, and to an increase in the utilization rate of capital. Beyond S, any further increase in the consumption is blocked by the capital constraint, and the only effect is a corresponding increase in unemployment.[31]

[31] Note how Rybczynski's theorem pertaining to an equilibrium without specialization holds in our setting. As x increases, the capital intensive consumption good increases and the investment good (with zero capital intensity) decreases. This conclusion is reversed with an increase of labour. For details, see for example Jones (1965a, 1965b), and also Johnson (1971).

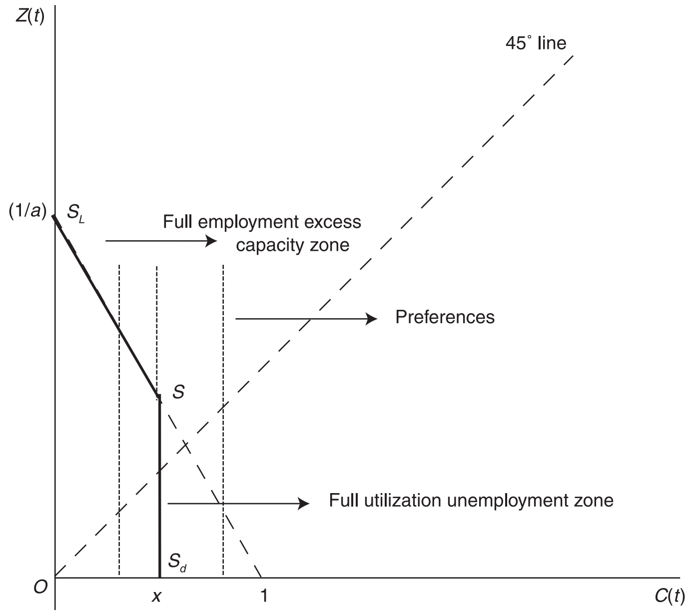

FIGURE 3. Production possibility surface and planner's preferences

It is now a simple matter to depict the planner's preferences in the (x, x') plane. By specification and construction, the amounts of labour and capital being used in the consumption good sector are both identical to the amount of the consumption good and the level of utility being produced, and thus the indifference curves are simply the iso-employment lines. The line *MV* now takes on a new identity having solely to do with preferences; it pegs a map of kinked (Leontief-type) indifference curve in the (x, x') plane. The planner's preferences are complete in the sense that we now have an indifference map over the entire transition set Ω with *OVL* marking the zero utility and *mMD* the maximum utility indifference curves. We shall denote the utility function[32] corresponding to this indifference map is by $(x, x') \rightarrow u(x, x')$.

The basics of our geometrical apparatus are now all laid out: the 45° line, the lines *VL* and *OD* constituting Ω, the point *M* corresponding to the (normalized) unit labour supply, and the line *MV* delineating both a privileged subset of Ω and the indifference map corresponding to u.[33]

4. Determination of the benchmarks

The first unknown to be determined is a standard benchmark in the theory of intertemporal resource allocation—the level of the capital stock that allows a maximal sustainable utility level, the so-called golden rule stock $\hat{x}$. McKenzie refers to this as a "von Neumann

[32] Analytically, the utility function is given by $u(x, x') = \max\{y \in \mathbb{R}_+ : 0 \leq x \text{ and } y \leq 1 - a(x' - (1 - d)x)\} = \min[1 - a(x' - (1 - d)x), x]$.

[33] The period to period production set Ω has been specified in footnote 25 and the utility function u in footnote 32.

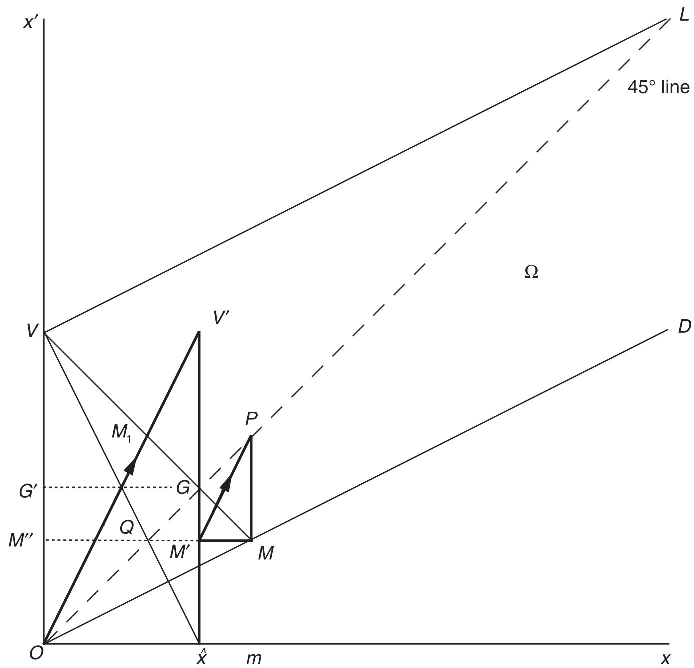

FIGURE 4. Determination of the golden rule stock, the golden rule price system and the corresponding no value loss line

point" and describes it as a "point of maximal sustainable utility, the terminal capital stocks must be as large as the initial stocks, and the utility must be as large as possible given this condition."[34]

Such a capital stock is easily seen in Figure 4 to be the unique one-period plan G obtained by the intersection of the 45° line with MV. But the geometry actually enables us to compute it once we recall the slope of the line MV identified in (3) above. Since $\angle VGG' = \angle VMM'$,

$$\xi = \frac{VG'}{G'G} = \frac{VM''}{M''M} \Rightarrow \frac{(1/a) - \hat{x}}{\hat{x}} = 1/a - (1-d) \Rightarrow \hat{x} = \frac{1}{1+ad}. \tag{5}$$

The golden rule stock is the solution to a maximization problem in which the objective function u is maximized over the constraint set Ω and the additional sustainability constraint that terminal stocks are not less than the initial one.[35] As such, there is a shadow price $\hat{p}$ associated with the sustainability constraint. On appealing to Uzawa's version of the Kuhn-Tucker theorem Uzawa (1958),[36] we can write

[34] See McKenzie (1999, p. 389). For a formal definition in the context of this model, see Khan and Mitra (2003), and more generally Gale (1967a), Brock (1970). In the sequel, depending on the context, we shall use the terms "golden rule stock" and "von Neumann point" interchangeably.

[35] Analytically, we maximize $u(x, x')$ subject to $x' \geq x$ for all $(x, x') \in \Omega$.

[36] Note that all of the conditions for the invoking of this theorem are satisfied: u is a concave function, Ω is a convex set, and the positivity of a leads to Slater's form of the Kuhn-Tucker qualification being satisfied. Also see Gale (1967b) for a generalization relevant to context of growth theory.

$$u(x, x') + \hat{p}(x' - x) \le u(\hat{x}, \hat{x}) \qquad \text{for all } (x, x') \in \Omega. \tag{6}$$

We can now follow Radner (1961) and define the value loss $\delta_{(\hat{p},\hat{x})}(x, x')$ at the golden rule price system $\hat{p}$ associated with the one-period plan (x, x') by rewriting the above as[37]

$$\delta_{(\hat{p},\hat{x})}(x, x') = u(\hat{x}, \hat{x}) - u(x, x') - \hat{p}(x' - x) \qquad \text{for all } (x, x') \in \Omega. \tag{7}$$

All this is a standard rehearsal of a key concept in the general theory of intertemporal resource allocation;[38] what is new is that we have the machinery we need to furnish a clear geometrical representation of this idea.

We begin with, the determination of the golden rule price system $\hat{p}$. Towards this end, consider Figure 4, and note that the zero net investment plan M, given by $(1, (1 - d))$, can be substituted in Equation (6) to yield

$$1 + \hat{p}(1 - d - 1) \le \hat{x} \Rightarrow \hat{p} \ge \frac{1 - \hat{x}}{d} \Rightarrow \hat{p} \ge \frac{M'M}{PM}. \tag{8}$$

By the same token, the maximal net investment, zero consumption one-period plan M, given by $(0, 1/a)$, can be substituted in Equation (4) to yield

$$0 + \hat{p}(1/a - 0) \le \hat{x} \Rightarrow \hat{p} \le \frac{\hat{x}}{(1/a)} \Rightarrow \hat{p} \le \frac{VV'}{OV}. \tag{9}$$

Now, on using the value of $\hat{x}$ in Equation (5), an easy computation yields

$$\frac{1 - \hat{x}}{d} = \frac{1 - \dfrac{1}{1 + ad}}{d} = \frac{ad}{d(1 + ad)} = \frac{\dfrac{1}{1 + ad}}{1/a} = \frac{\hat{x}}{(1/a)}.$$

Hence, the weak inequalities are all equalities in the following expression,

$$\frac{M'M}{PM} = \frac{1 - \hat{x}}{d} \le \hat{p} \le \frac{\hat{x}}{(1/a)} = \frac{GG'}{VO}. \tag{10}$$

But then in terms of the geometrical development, we have shown that the angle $\angle OV'G$ equals the angle $\angle M'PM$, which implies that the line $V'O$ is parallel to the PM', and hence that the slope of either of the lines $V'O$ and PM', measured relative to the Y-axis, is the golden rule price system.

All that remains is the determination of the zero value loss line, which is to say, the locus of all plans for which $\delta_{(\hat{p},\hat{x})}(x, x') = 0$. From Equation (7), we obtain

---

[37] Note that we are defining a function $\delta_{(\hat{p},\hat{x})}: \Omega \to \mathbb{R}_+$. We depart from the literature in retaining the subscript $(\hat{p}, \hat{x})$ in $\delta_{(\hat{p},\hat{x})}(x, x')$ even though the golden rule price system will remain fixed in the sequel.

[38] As surveyed for example in McKenzie (1986). One may quote here Gale's (1967, p. 22) statement that the necessity and sufficiency of Brock (1970) for the $\hat{x}$ to be the golden rule stock "provides the single most important tool in modern economic analysis both from the theoretical and computational point of view."

$$\begin{aligned}\delta_{(\hat{p},\hat{x})}(x, x') &= u(\hat{x}, \hat{x}) - u(x, x') - \hat{p}(x' - x) \\ &= [u(\hat{x}, \hat{x}) - x - \hat{p}(x' - x)] + [x - u(x, x')] \\ &= \text{the shortfall from golden rule utility level + idle capacity.} \end{aligned} \tag{11}$$

The first point to be noticed in this connection is that the absolute value of the slope of the line *MV*, ξ, also turns out to be characterized by the golden rule price system in a particularly simple form. Working with Figure 4, we obtain

$$\xi = \frac{VG'}{G'G} = \frac{VO - G'O}{G'G} = \frac{1 - \dfrac{G'O}{VO}}{\dfrac{GG'}{VO}} = \frac{1 - \hat{p}}{\hat{p}}, \tag{12}$$

where the last equality follows from the above identification of the golden rule price system. Once the slope of the line *MV* is determined, we need only its intercept for a full identification. Since *MV* passes through the golden rule stock point $\hat{x}$ on the 45° line, we obtain its equation as

$$x' = -\frac{1 - \hat{p}}{\hat{p}}x + C \Rightarrow \hat{x} = -\frac{1 - \hat{p}}{\hat{p}}\hat{x} + C \Rightarrow \hat{p}x' + (1 - \hat{p})x = \hat{x} = u(\hat{x}, \hat{x}). \tag{13}$$

We have already seen the line *MV* serving three distinct roles: as the full employment line; no excess-capacity line; and as a benchmark for delineating the iso-employment lines in the consumption sector as well for the planner's preferences. The latter yielded, in particular, the conclusion that the utility level, $u(x, x')$, of any plan (x, x') on *MV* is x. On substituting Equation (13) in (11), we obtain

$$\hat{p}x' + x - \hat{p}x = u(\hat{x}, \hat{x}) \Rightarrow 0 = u(\hat{x}, \hat{x}) - u(x, x') - \hat{p}(x' - x) = \delta_{(\hat{p},\hat{x})}(x, x') \tag{14}$$

and thereby discover a fourth identity of the line *MV* in this geometric development: it is the zero value loss line, and constitutes what is referred to as the von Neumann facet.[39]

But now, we are in a position, through Figure 5, to determine the value loss of any plan in Ω. As Equation (11) makes clear, this value loss is not determined solely by the difference in intercepts (say, on the 45° line) of *MV* and a line parallel to it and passing through the one-period production plan. For one thing, this would lead us to conclude that there is value gain as the line *MV* moves outwards. Lines parallel to *MV* are indeed iso-value loss lines, but they depict the value loss after taking excess capacity into account. For all plans, say *F* with coordinates (x_0, x_0'), in the surplus labour triangle *MOV*, there is no excess capacity of capital and hence its utility is furnished by its first coordinate, leading to the second term in Equation (11) being zero. Hence its value loss consists only of its shortfall from the golden rule utility level, the first term in (11). This is given by the difference between $\hat{x}$ and the abscissa of the point of intersection of the 45° line with a line *M′V′*

[39] See McKenzie (1986, 1987, 1999), McKenzie (1990, p. 391) writes "The von-Neumann facet plays a crucial role in the multi-sector Ramsey model once the assumption of strict concavity of the reduced utility function is dropped." As the reader has noticed, this assumption of strict concavity does not hold for the special case of the RSS model that we are working with; also see Nishimura and Yano (1995, p. 994; 2000, p. 264).

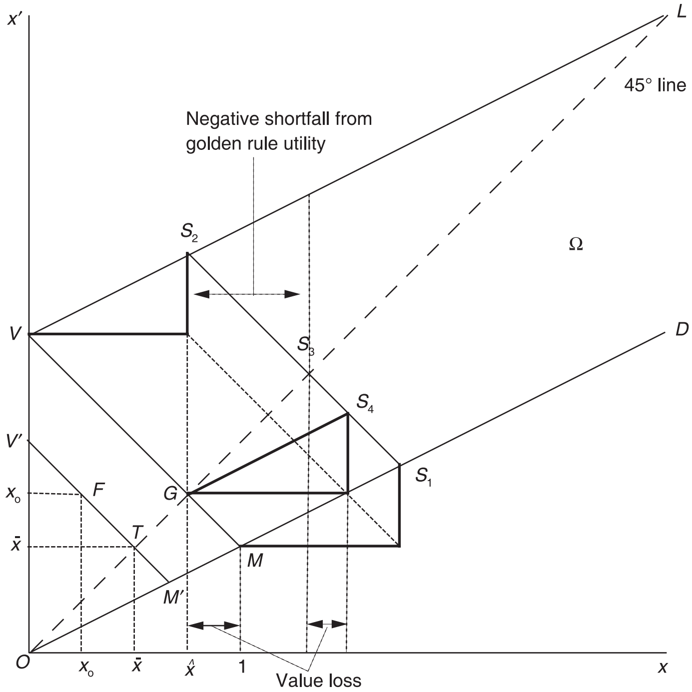

FIGURE 5. Iso-value loss lines

parallel to MV and passing through F. Let the coordinates of this point of intersection T be given by $(\bar{x}, \bar{x})$, and hence the equation of the line $M'V'$ is given by[40]

$$x' = -\frac{1-\hat{p}}{\hat{p}}x + C \Rightarrow x' = -\frac{1-\hat{p}}{\hat{p}}x + \frac{\bar{x}}{\hat{p}}. \tag{15}$$

We can now obtain the shortfall from the golden rule utility level that we seek:

$$\begin{aligned}\frac{\bar{x}}{\hat{p}} = \left(x_0' + \frac{1-\hat{p}}{\hat{p}}x_0\right) &\Rightarrow u(\hat{x}, \hat{x}) - \bar{x} = u(\hat{x}, \hat{x}) - \hat{p}x_0' - (1-\hat{p})x_0 \\ &= u(\hat{x}, \hat{x}) - x_0 - \hat{p}(x_0' - x_0) = \delta_{(\hat{p},\hat{x})}(x_0, x').\end{aligned} \tag{16}$$

In this demonstration, we have also shown that any plan on $M'V'$ has the same value loss.

Next, we turn to plans in the "open" parallelogram $LV\,MD$. In this case, value loss stems from both excess capacity and from the negative shortfall from the golden rule utility level. We have already seen that this shortfall is the same for all plans on the line S_1S_2 parallel to MV, and is given by the difference between $\hat{x}$ and the abscissa of the point of intersection S_3 of S_1S_2 and the 45° line. In order to show that S_1S_2 is an iso-value loss line, all that remains for us to show is that the excess capacity associated with any plan on it,

[40] The procedure is identical to that already used in Gale (1967a) in the derivation of the equation of the line MV.

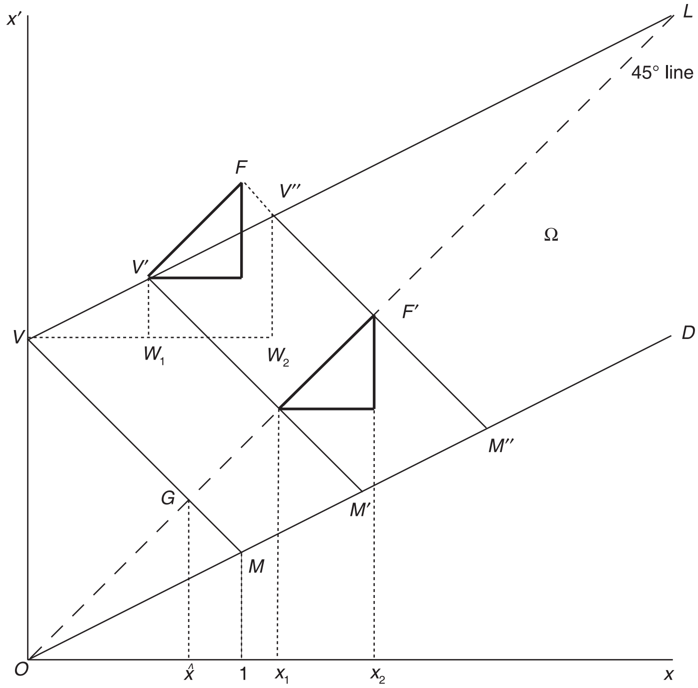

FIGURE 6. Changes in value loss as VM moves outwards

say S_2, S_3, S_4 or S_1, is identical. This is easy from our procedure for computing excess capacity: all of the triangles with vertices S_1, S_2 and S_4 exhibited in Figure 5 are congruent, and hence their bases are equal.

Next, we show that the value losses increase as iso-value loss lines move "away" from the zero value loss line MV in either direction. This is clear when we limit ourselves to the full capacity, surplus labour triangle OMV. In the full employment, excess capacity area $LVMD$, the difficulty lies in the fact that as MV moves outwards, both the negative shortfall from golden rule utility as well as the excess capacity increase. However, the latter increases more than the former. To see this, consider the parallel lines $M'V'$ and $M''V''$ in Figure 6. The increase in the shortfall amounts to x_1x_2, whereas the increase in the excess capacity is the amount W_1W_2. To see that W_1W_2 is always greater than x_1x_2, draw a line $V'F$ parallel to the 45° line, and observe that the difference in the abscissae of the points F and V' (which is x_1x_2 since triangles with vertices F and F' are congruent) is smaller than W_1W_2. And this is always so by virtue of the fact that the slope of the 45° line is steeper than the slope of OD, which is another way of saying that the rate of depreciation d is always less than unity.

Finally, we show that the sum of the value losses of two plans, say P_1 and P_2 on a horizontal line beginning at G as in Figure 7, equal the value loss of their sum $P_3 = P_1 + P_2$. In Figure 7, let H_1P_1, H_2P_2 and H_3P_3 be iso-value loss lines. Since GP_1 equals P_2P_3 by hypothesis, the triangles GH_1P_1 and $H_2H_3Z_1$ are congruent. Hence GH_1 equals H_2H_3 and hence H_2Z equals GS_1. This establishes that $GS_1 + GS_2 = S_2S_3 + GS_2 = GS_3$, and hence the sum of the individual shortfalls from golden rule utility is the shortfall of the sum. Next, note the congruence of the triangles GG_1P_1 and $G_2G_3Z_2$, and hence the equality of GG_1 and G_2G_3, and therefore of their projections on the horizontal through G and G_2 respectively. Hence

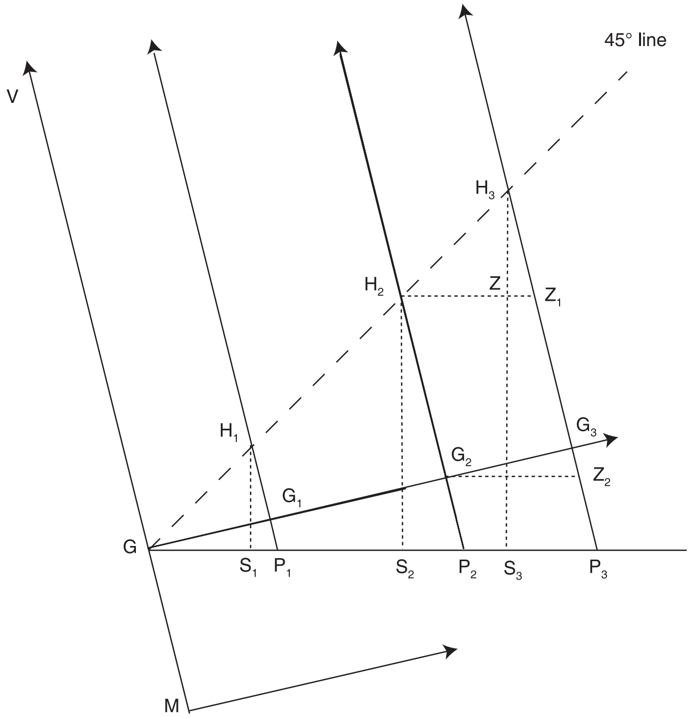

FIGURE 7. Sum of value losses equals the value losses of the sum

the sum of the excess capacity at G_1 and G_2 equals the excess capacity at G_3. This completes the proof of our claim.

In iso-value loss lines, we have a basic geometric tool to characterize optimal policies and we turn to providing such a characterization.

5. Optimal policies ($-1 < \xi \leq 0$): monotonic convergence

We begin with Figure 8 which depicts the case where the slope of the *MV* line is positive and lies in the open unit interval, i.e., where $-1 < \xi < 0$. Since the geometry developed is premised on this slope being negative, this may be disorienting, but we invite the reader to check that all of the previous arguments carry through in a routine manner.[41] However, for economy of notation and expression, we shall henceforth refer to the golden rule stock as g, and the unit capital stock as m. We can now read off the singular values $m_1^r, m_2^r, \ldots$ to the right of m, and $g_1^r, g_2^r, \ldots$ to the right of g.[42]

[41] This is also a good exercise; geometrical argument is perhaps even less of a spectator sport that analytical argument.

[42] This is simply to say that a countable number of intervals (m_n^r, m_{n+1}^r), $n = 0, 1, 2, \ldots,$ $m_0^r = $ m, cover the interval $x > m$, where the relevant formula is $m_n^r = 1/(1-d)^n$, $n = 0, 1, 2, \ldots$. Similarly, a countable number of intervals (g_n^r, g_{n+1}^r), $n = 0, 1, 2, \ldots,$ $g_0^r = g$, cover the interval $g \leq x < m$, where the relevant formula is $g_n^r = $ g$/(1-d)^n$, $n = 0, 1, \ldots$.

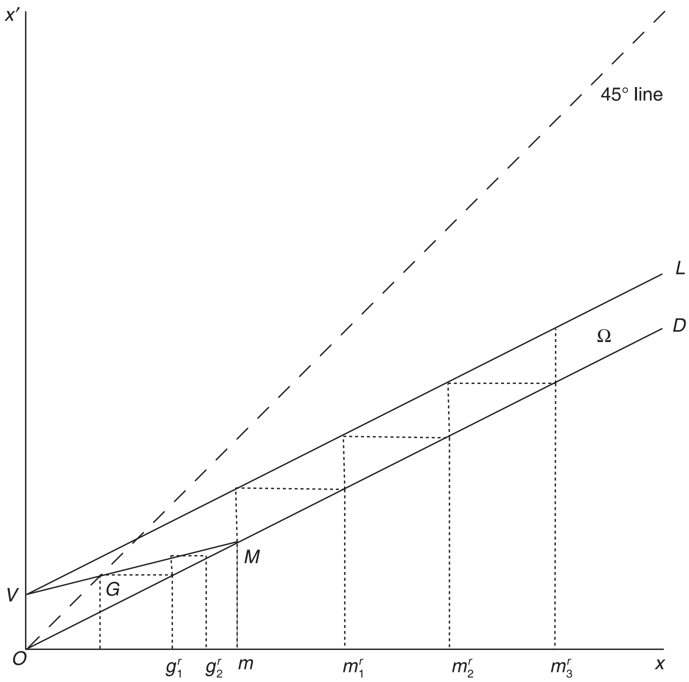

FIGURE 8. Transition dynamics $-1 < \xi < 0$

We can now claim that for any initial capital stock in the interval (0, m), the optimal policy is to choose a plan on MV. It is easy to see that the programme obtained in this way monotonically converges to the golden rule stock, and that it has an aggregate value loss of zero. Thus, an appeal to the extension of Brock's theorem (presented as Theorem 1 in the Appendix) establishes the claim. This result can be succinctly but informally stated as the assertion that, for the case under consideration, any optimal programme starting on the von Neumann facet stays on the facet and thereby converges to the von Neumann point.[43]

It is now easy to characterize the policy function globally. For any initial capital stock, say in the $(n+1)^{th}$ interval (m_n^r, m_{n+1}^r), $n \geq 0$, the optimal policy is to choose the next period's capital stock to be on the line MD and thereby move to the n^{th} interval in the next period. The resulting programme will converge to the von Neumann facet in $n + 1$ periods, and by construction, it is one that minimizes aggregate value loss among all programmes starting from the same initial stock. Hence it is an optimal programme. The optimal policy then is to follow the path charted by the two lines MV and MD, which is to say, the path tracked by the difference equation,

$$x(t+1) = \max[-\xi x(t) + 1/a, (1-d)x(t)] \qquad \text{for all } t = 0, 1, \ldots. \tag{17}$$

[43] For emphasis, in this statement, we refer to the golden rule stock as the von Neumann point; see McKenzie (1968). Note also that we are now referring to a programme "starting on the von Neumann facet" by which we mean that it starts from an initial stock in the projection to the space of initial stocks (the X-axis) of the von Neumann facet rather than the facet itself. We trust that the abbreviated expression will not be confusing to the reader.

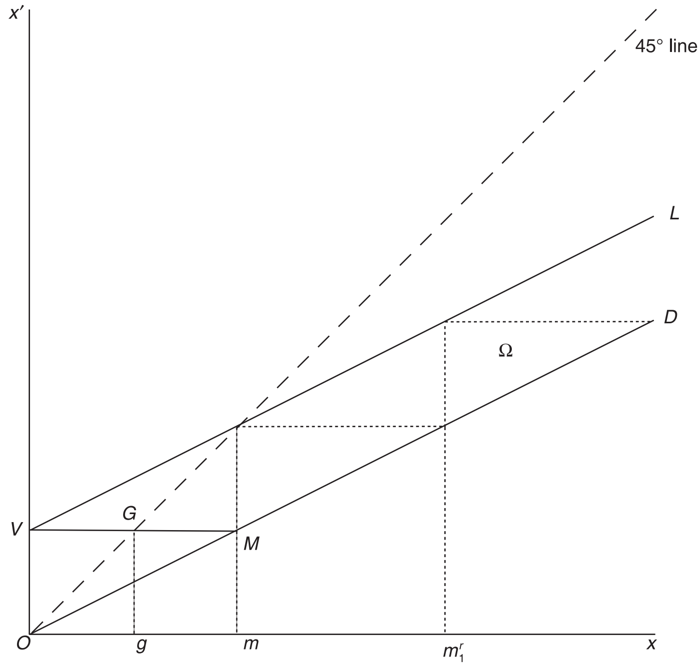

FIGURE 9. Transition dynamics $\xi = 0$

There is only one additional point to be made for the analysis to be complete. This is the observation that for the initial capital stocks g_n^r, $n = 0, 1, 2, \ldots$, there is convergence to the von Neumann point (golden rule stock) rather than to the facet, in a finite number of periods. These possibilities are open neither to a capital poor economy, nor to an economy which is capital rich in the sense of having an initial capital stock greater than, or equal to, unity. This is a rare (accidental) event of history. This is simply to say that the g_n^r are of the Lebesgue measure zero.

Next, we turn to the case where the *MV* line is horizontal, i.e. $\xi = 0$, which implies that $(1 - d) = 1/a = 1/(1 + ad) = \hat{x}$. Figure 9 tells the complete story. The optimal programme converges to the von Neumann facet *MV* in a finite number of periods, and to be on the von Neumann facet is to be at the golden rule stock in the subsequent period. A distinguishing characteristic of this case is that the singular points g_{n+1}^r and m_n^r coincide. The optimal policy is to follow the path tracked by the difference equation,

$$x(t + 1) = \max[1/a, (1 - d)x(t)] = \max[\hat{x}, (1 - d)x(t)], \qquad \text{for all } t = 0, 1, \ldots, \qquad (18)$$

which is simply a special case of Equation (17) with $\xi = 0$.

6. Optimal policies ($0 < \xi < 1$): damped cyclical convergence

The fact that optimality requires any programme starting on the von Neumann facet to stay on the facet is a special property of the two cases considered so far. In particular, as can be checked from considering the plan *V* in Figure 10, this is no longer necessarily so when the slope of the *MV* line is negative and between zero and one in absolute value; i.e.

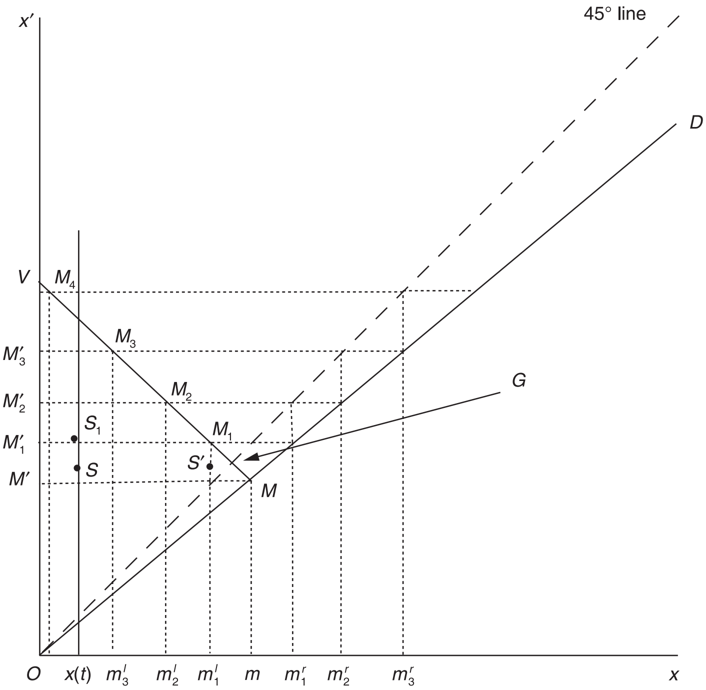

FIGURE 10. Transition dynamics $0 < \xi < 1$

$0 < \xi < 1$. Here, we can use the values $m_1^r, m_2^r, \ldots$ to read off the singular values $m_1^l, m_2^l, \ldots$ to the left of m. Since $(0, m)$ is a finite interval, there will only be a finite number, say $\bar{n}$ of such values.[44] In Figure 10, $\bar{n}$ equals one.

We can now utilize the methods of the previous section to conclude that the optimal policy function for any initial capital stock greater than or equal to m_1^l is to follow the path charted by the two lines M_1M and MD, and as such, it is an interesting mix of monotonic and cyclically damped outcomes. For all initial stocks greater than m, the optimal policy is just as before; namely, to choose a corresponding plan on the line MD and move to the adjacent left interval. This is to say that an optimal policy for a capital rich economy is to depreciate down to the von Neumann facet. Once on this facet, the optimal policy is to converge to the golden rule stock, but unlike the previous cases, it is to proceed to it cyclically—by over-building and by under-depreciating relative to the golden rule stock (von Neumann point). The only exception to a cyclical outcome is represented by the singular values g_n^r (not shown in Figure 10 to avoid clutter) which lead to monotonic convergence.

We now see that by virtue of m_n^l being undefined in Figures 8 and 9, the property that characterized the von Neumann facet now applies only to a distinguished subset of it, namely M_1M. We shall refer to it as the McKenzie facet; it includes the von Neumann point and has the property that any programme starting on it remains there. The question still to be answered is what the planner ought to do regarding initial stocks that are on the

[44] Formally, this is simply to say that there exists $\bar{n}$ such that a finite number of intervals (m_{n+1}^l, m_n^l), $n = 0, 1, 2, \bar{n}$, and $(m_n^l, 0)$ cover the interval $(0, m)$ where m_0^l is also m. The relevant formula is $m_n^l = (1/\xi)((1/a) - m_n^r) = (1/\xi)((1/a) - 1/(1 - d)^n)$, $n = 0, 1, \ldots, \bar{n}$.

von Neumann facet but not on the McKenzie facet. Put differently, we need give a characterization of the policy function that is global and includes the interval $(0, m_1^l)$.

We can now assert that the policy function is given by Equation (17). This is to add to our previous claims the observation that for any initial capital stock in the interval $(0, m_1^l)$, the planner chooses a corresponding plan on the line segment VM_1 not including M_1. Suppose this is not the case and, to begin with, we have a situation pictured in Figure 11a whereby, along the optimal programme, S is chosen instead of the the full employment plan F, followed a period later by S'. We can show that an alternative programme with full employment in the period under consideration but otherwise lagged to the given one by one-period, yields a lower aggregate value loss. This is to argue that the value loss at S is strictly higher than at S'. This can be easily demonstrated. Note that TU represents the value loss at S. Let U' be the intersection of the vertical through F' and a line parallel to the 45° line through U. Let T' be the intersection of the 45° line and the horizontal through U'. It is clear that the triangles TUG and $T'U'F'$ are congruent. Hence $T'F'$ equals TG equals SW, where the line SW is parallel to the 45° line. Since SF equals $S'F'$ by hypothesis, the triangle SFW is congruent to the triangle $T'S'F'$, and hence $\angle SFW$ equals $\angle F'S'T'$, allowing us to conclude that $S'T'$ is parallel to MV and hence is an iso-value loss line. Since the rate of depreciation is less than unity, the line through G parallel to OD intersects $T'S'$ at a point to the right of T', and hence the value loss at S' is strictly less than at S. Note that the success of this demonstration hinges crucially on the discount factor being unity, but not on the slope of the line MV.[45]

Thus the only question is whether the alternative programme of the type that we considered above in Figure 11 is available to the planner. Going back to Figure 10, we see that for the stock $x(t)$, any plan whose abscissa is below S_1 such as S, precludes the availability of such a programme. But in this case, a four-fold argument delivers the conclusion that we seek. Consider a programme starting at $x(t)$ above MV. Given what we have concluded about the optimal policy for plans on MD, an alternative programme starting on MV but otherwise identical to the given plan yields a lower value loss. For a programme starting from a plan equal to or above $M_1'M_1$ but below MV, the type of programme that we charted through the consideration of Figure 11 above is indeed available and the previous argument applies. All programmes starting at plans with ordinates below M have a higher aggregate value loss than a programme that reaches the golden rule stock in one period. Thus, we need to concentrate only on the fourth possibility, concerning programmes starting at a plan whose abscissa is $x(t)$ and whose ordinate is between those of M and M_1, say S. Note that the aggregate value loss of any programme starting at S_1 is the value loss at S_1 plus the aggregate value loss of any programme starting at M_1. Similarly, the aggregate value loss of any programme starting at S is the aggregate value loss of any programme starting at S' plus the difference in the value losses of the plans S and S'. Since S_1M_1 equals the interval SS', the value loss at S_1 equals the difference in the value losses of the plans S and S'. Then, if the aggregate value loss of the programme starting at S is less than or equal to that starting at S_1, the aggregate value loss of a programme starting at S' is less than or equal to that starting at M_1; but this is a contradiction.

So we have now extended the optimal policy function from the union of MD and MM_1 to the union of MD and MM_2. We now proceed to the next left adjacent interval to (m, m_1^l),

45 In order to show that the argument applies without any change, Figures 11b and 11c reproduce Figure 11a with different absolute values of the slope of MV.

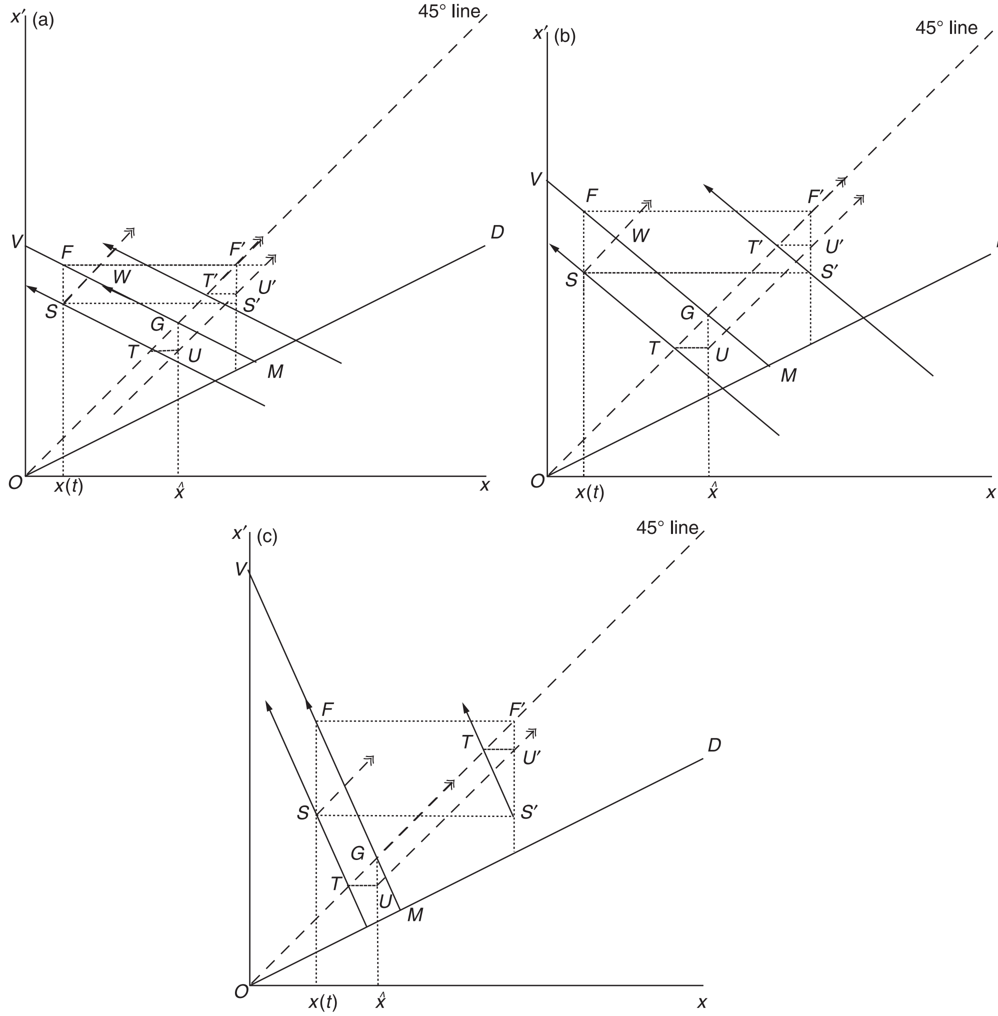

FIGURE 11. (a) Full employment $0 < \xi < 1$; (b) Full employment $\xi = 1$; (c) Full employment $\xi > 1$

and repeat the argument with $M_2'M_2$ playing the role of $M_1'M_1$. Since there are a finite number of left intervals, we obtain an optimal policy function for any non-negative initial capital stock. The argument is complete.

There is only one additional point to be made. This concerns the singular initial capital stocks g_n^r, $n = 0, 1, 2, \ldots$, and g_n^l, $n = 0, \ldots, \bar{n}$, from which there is convergence to the golden rule stock in a finite number of periods.[46] For a capital rich economy, one that finds itself at g_n^r, this convergence is monotonic. These possibilities are also available to a capital poor economy, but only after a (heavy) over-building phase in the first period, and the poorer the economy, the heavier the phase!

[46] To avoid clutter, we do not mark them in Figure 10. The formula for g_n^l remains to be furnished, but as in footnotes 42 and 44, $g_n^l = (1/\xi)((1/a) - g_n^r) = (1/\xi)((1/a) - g/(1 - d)^n)$, $n = 0, 1, \ldots, \hat{n}$. Now $g_0^r = g_0^l = g$.

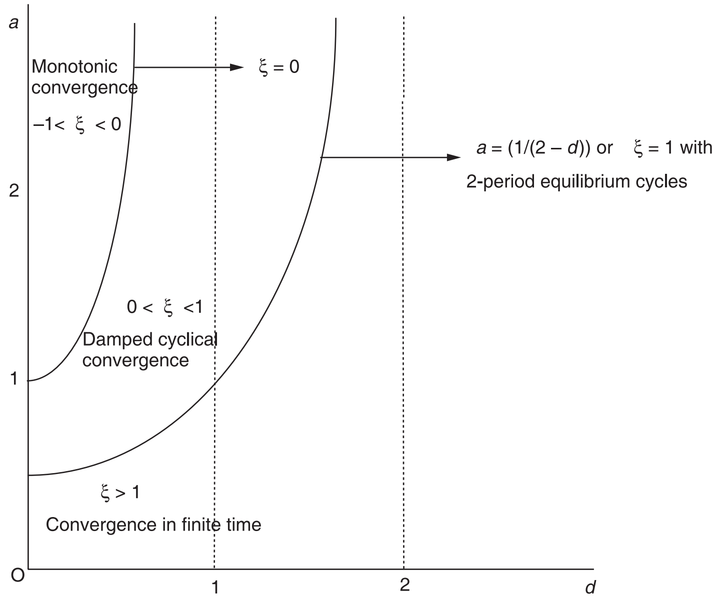

FIGURE 12. Parameter specifications

7. Optimal policies ($\xi = 1$): convergence to limit cycles

In Figure 4, the line *MV* is orthogonal to the 45° line, an incidental aspect that we have so far ignored in our discussion. This orthogonality is simply a consequence of how the two parameters, *a* and *d*, relate to each other. In particular, from Equation (3), and the fact that the $\angle OVM$ is 45°,

$$\xi = 1/a - (1-d) = \frac{VM''}{M''M} = 1 \Rightarrow a = \frac{1}{2-d}. \tag{19}$$

An alternative way of seeing this is to note

$$(1/a) = OV = VM'' + M''O = M''M + M''O = 1 + (1-d) = 2 - d.$$

For later reference we graph this relationship in Figure 12, and simply note here that it depicts the range of parameter values of the model, albeit singular, under which we work in this section.[47]

Note that, irrespective of the slope of *MV*, the segment *PM* (say in Figure 4) equals the rate of depreciation *d*. In Figure 13, PM_1 and *QM* are parallel lines measuring the ordinates of the points *P* and *M* respectively. Since *MV* has a slope of 45°, M_1PM and *PMQ* are isosceles triangles, and hence PM_1 and *QM* are also equal to *d*. Hence M_1Q is a vertical line segment, and therefore also of length *d*. In other words, $PMQM_1$ is a square with the golden rule stock as its centre (indicated as *G* in Figure 4, but not indicated in Figure 13).

[47] Note also the function relating *a* to *d* is an infinitely differentiable convex function.

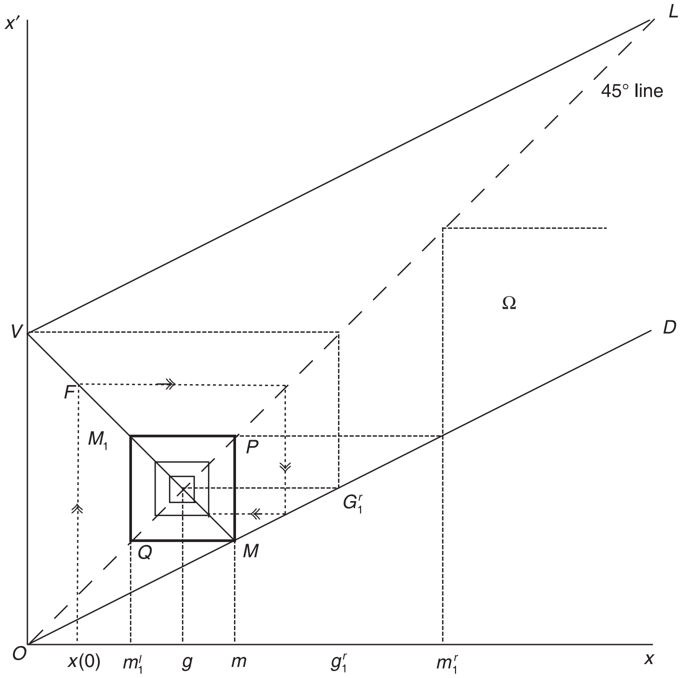

FIGURE 13. Transition dynamics $\xi = 1$

With this, the geometry brings to light interesting invariants pertaining to the case that we are considering. Irrespective of how a and d change, but provided that they continue to relate to each other as graphed in Figure 12, the golden rule stock $\hat{x}$ is given by $1 - V'V = 1 - (d/2)$, the abscissa of Q is $(1 - d)$, and finally, the golden rule price system, $\hat{p}$, as given by the ratio, $M'M$ to PM, is always one half, and thereby independent of a and d. Since OG equals GV (in Figure 4), the golden rule stock $\hat{x}$ could alternatively be expressed as $1/2a$.[48] Finally, note that the slope of the lines OM_1 and OV' also equal one half, and hence the points O, M_1 and V' are collinear. Similarly, the points V, Q and g are collinear.[49]

We can now use the methods of the previous section to assert that MM_1 is the McKenzie facet, any programme starting on it remains on it, and since it has an aggregate value loss of zero, it is an optimal programme. This is a result that also has a surprise to it, ruling out as it does, other plausible intuitions about the optimal policy. Thus, given the initial capital stock in the interval (g, m_1^l), it rules as non-optimal the entirely feasible trajectory of maintaining unemployment and "jumping" to the golden rule stock the next period—to be at the mid-point of M_1Q rather than at M_1. Again, with an initial capital stock of unity, it rules as non-optimal the policy of maintaining consumption at the golden rule level at the level and investing so as to have the golden rule stock of machines tomorrow—to be at the mid-point of PM rather than at M.

48 Once the geometry allows us to see these invariants, it is a simple matter to obtain them algebraically from the formulae furnished above.

49 Since the points M_1 and Q' are obtained in different ways in Figures 4 and 13, the last two observations show that there is no inconsistency.

An important aspect of these optimal programmes deserves to be highlighted. The geometry, in particular the fact that $PMQM_1$ is a square with the plan (g, g) as its centre, makes it evident that the average of the optimal programme starting from any initial stock in the interval QV is precisely the golden rule stock. This is Brock's average turnpike property which asserts that the average of the capital stocks of all "good programmes" (and not only the optimal ones) converge to the golden rule stock.[50] This is precisely the case when the optimum programme remains on the von Neumann facet but does not converge to the von Neumann point, as is the case here.[51]

When we turn to initial capital stocks that lie outside the interval MM_1, we fall back on the methods of the previous section to show that choices of plans on the union of the lines MV and OD lead to a programme that minimizes aggregate value losses of all programmes starting from the same initial stock, and hence is optimal. As in Section 5, we first consider intervals to the right of m, and then utilize the methods of Section 6 to consider intervals to the left of m.[52] Within these intervals we identify the singular values g_n^r and g_n^l from which the optimal programme converges to the golden rule stock in a finite number of periods. Indeed, if we regard the golden rule stock as an equilibrium cycle with a zero amplitude, we can read off singular values corresponding to any particular equilibrium cycle such that the optimal programme from these identified values converges to that particular cycle in a finite number of periods. Whereas for any particular cycle, these singular values are "few", their union over the uncountable infinity of equilibrium cycles will cover the entire (non-negative) X-axis.

We are now in a position to conclude this section with two observations. First, note that the set of equilibria to which the optimal programmes converge are of uncountable cardinality; and whereas it does not make sense to say that this cardinality is inversely related to the rate of depreciation, we can say that the "size" of the McKenzie facet, the length of the interval MM_1 serving as a proxy for this size, increases as we decrease both d and a in a way that is consistent with the relationship pictured in Figure 12. The second point relates to what is termed as "history independent" dynamical systems in Mitra and Nishimura (2001b). The optimal policy function underlying the optimal programme leads to a dynamical system which is anything but history independent. The speed of convergence to the equilibrium, but perhaps more importantly, the particular equilibrium and its amplitude, are all dependent on the initial stock of machines the economy is endowed with. However, if we use the amplitude of the equilibrium cycle as a measure of the "volatility" of equilibrium, our reference to m_n^l, g_n^l and to m_n^r, g_n^r as singular values is justified. They represent values of the initial capital stock at which volatility of equilibrium is respectively maximized and minimized. This is illustrated in Figure 13.

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50 See Gale (1967a) for a definition of "good programmes", and Brock (1970) for details pertaining to the "average turnpike" property. In particular, note the importance of the assumption that the golden rule stock is unique. In the context of the RSS model, see Khan and Mitra (2003), and in particular, Proposition 4 for the claim that any programme which is not good is "bad" in the sense of making infinite aggregate value losses. It is also worth pointing out that "good" programmes are referred to as "eligible" programmes in Koopmans (1965).

51 It is also the case when Inada's 1964 assumption that "all paths that remain on the facet forever converge uniformly to a maximal stationary path" does not hold; see MeKenzie (1987, p. 15) for a discussion.

52 Figure 11b and Footnote 45 are the relevant aids.

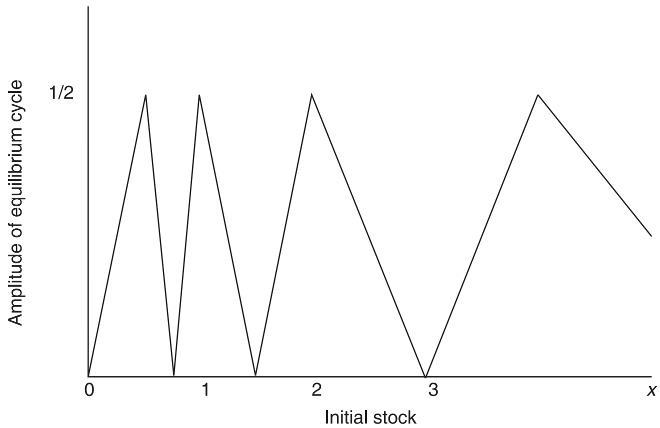

FIGURE 14. Volatility of equilibrium $\xi = 1$, $d = 1/2$

8. Optimal policies ($\xi > 1$): convergence in two periods

When we turn to Figure 15a which depicts the case where the slope of *MV* is greater than that of the 45° line in absolute value and $\xi > 1$, it is clear that, unlike earlier sections, the optimal policy is not tracked by the locus *VMD*. To see this, simply note that a programme starting at *M* cycles around the golden rule stock *G* in a way that the average turnpike property is violated and the programme keeps on accumulating value losses without bound. As such, it is bad.[53] Thus optimality requires that the planner jump off the locus *VMD*, and the question is when and to where. From earlier analysis of the RSS model, we know that an optimal programme must end at the golden rule stock, and once this is taken into account, the first part of the question admits the only possible answer, namely, "as soon as possible."[54] However, given irreversibility of capital, it takes time for the economy to dispose of its extraneous capital, and the convergence to the golden rule cannot be accomplished in one period. Furthermore, given the desirability of full employment, a capital poor economy over-builds, and subsequently over-depreciates relative to the maintenance of the golden rule levels, and this too postpones convergence.

A substantiation of these intuitions requires a more intricate geometry than we have so far encountered. To begin with, we have first to establish the desirability of full employment for a programme with a minimum aggregate value loss. In Section 5, we saw this as a consequence of optimality, and in Section 6, by a direct argument based on Figure 11a. As in Section 7, we leave it to the reader to check that the argument carries over to the present case.[55] However, we now work with the intervals (g_n^l, g_{n+1}^l) rather than the intervals (m_n^l, m_{n+1}^l). It is to be emphasized that we establish this result only for the *GM* segment of the *MV* line, and that there is no presumption as to optimality but simply of aggregate

[53] See Footnote 50 for a formal definition of a bad programme. This is just a graphical depiction of example 2 in Khan and Mitra (2003).

[54] For the long-run properties of an optimal programme, see Khan and Mitra (2003, Section 5).

[55] Figure 11c and Footnote 45 are the relevant aids.

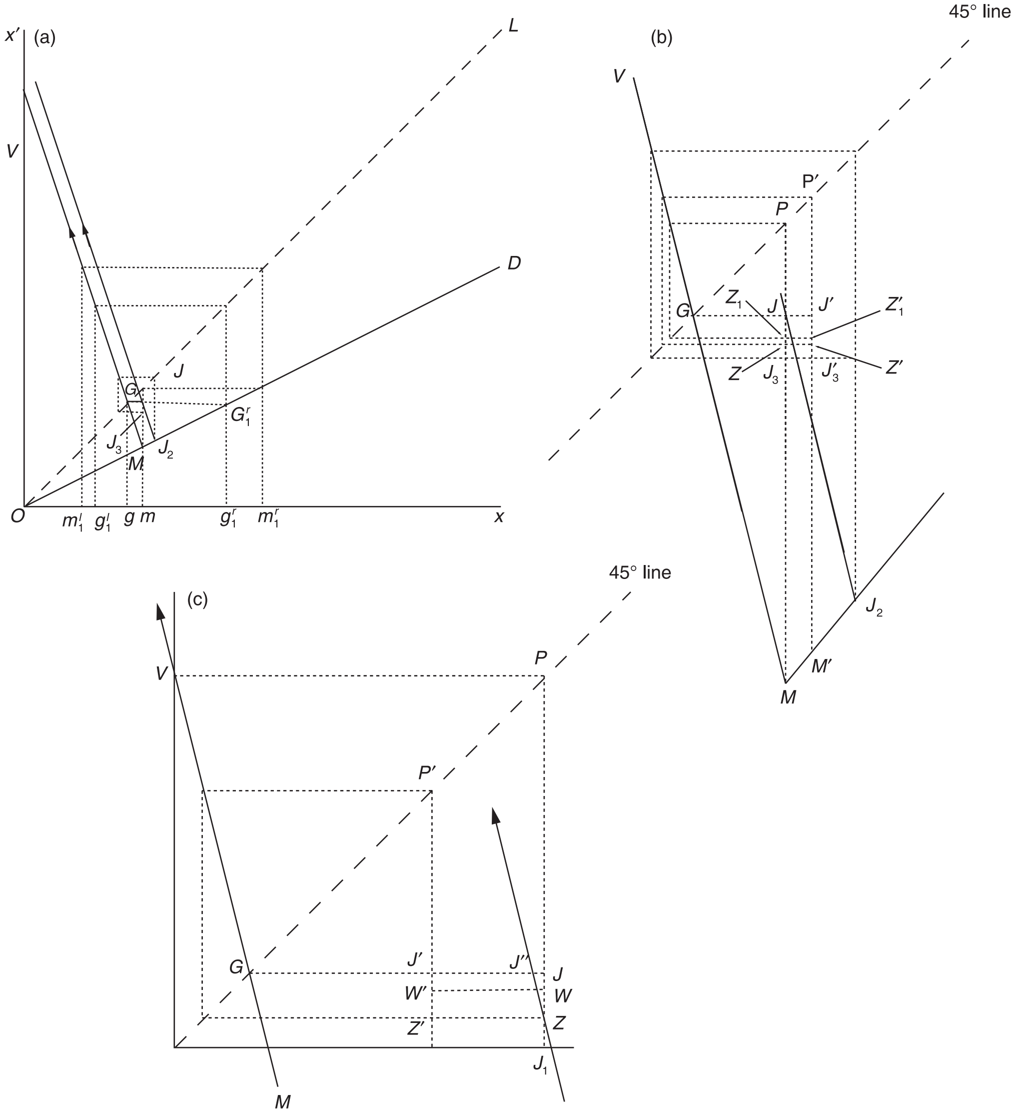

FIGURE 15. Transition dynamics $\xi > 1$

minimum value loss. With it, however, we have now all that is necessary to show that the optimal policy is to follow the path charted by the three lines VG, GG_1^r and $G_1^r D$ in Figure 15a. This is to say that it is tracked by the following difference equation,

$$x(t+1) = \max[-\xi x(t) + 1/a, \hat{x}, (1-d)x(t)] \qquad \text{for all } t = 0, 1, \ldots. \qquad (20)$$

The McKenzie facet is now claimed to shrink to the von-Neumann point and an optimum programme starting on GG_1^r "jumps" to G in one period and stays there. We shall establish this claim in a series of steps; we begin with an initial capital stock of unity, show how the argument applies to any initial stock in the interval (g, 1), and finally, by moving from interval to interval, as in the sections above, the entire policy function can be delineated.

In keeping with the first step, we work on the vertical through J in Figure 15a, and corresponding to J, construct the three benchmark plans J_1, J_2 and J_3 on it. J_1, not shown in Figure 15a, is the plan that leads to J next period if a policy of full employment is followed, J_2 is a plan on OD that has the same value loss as J, and finally, J_3 is the plan that leads to J_2 next period if a policy of full employment is followed. It is clear that the ordinates of J, J_1 and J_3 are strictly decreasing.[56] The fact that a programme that minimizes aggregate value loss among all programmes starting from a unit capital stock will not be on the segment above J and below J_3 is straightforward. Thus we need only consider the segment J_3J, obviously not including J.

Figure 15b magnifies Figure 15a around the von Neumann point G. We shall now proceed in two subcases depending on whether a plan is chosen in the segment J_3J_1 or the segment between J and J_1. Figure 15b is devoted to the first case. Suppose the planner chose J_1. Since she can also choose it next period, and since this was her optimal choice this period, stationarity of our model dictates that it is also her optimal choice next period. But this leads to a programme that keeps accumulating value losses ad infinitum and is not even a good programme,[57] let alone an optimal one. Thus, suppose that the planner's choice is represented by the plan Z strictly between J_1 and J_3 this period, and therefore limited to $P'M'$ next period. Since $P'M'$ lies strictly to the right of PM, the planner is limited to the segment between J' and J_3'. Suppose it to be Z_1' not equal to Z'. Given the stationarity of the model, the point Z_1 on PM is feasible, and the optimal path from Z_1' could have been followed one period earlier with a lower welfare loss by choosing Z_1 instead of Z. Thus, all that remains is a consideration of the choice of the point Z'. But if Z' is optimal next period, it will remain optimal in all subsequent periods, and we obtain a path that is not even good, let alone optimal.[58]

Thus, the choice of any plan on the segments J_1M and PJ on the line PM is not optimal. We now turn to the segment between J and J_1. This is more interesting than the segment J_1J_3 since the choice of any point on it leads, again through the optimality of full employment, to a capital stock next period lower than that of the current period, and hence the argument above does not apply.[59] In Figure 15c, a simplification of Figure 15b, let the planner's choice be represented by Z and therefore limited to the vertical through J' next period. Indeed, by an application of the previous argument, the planner is limited to the segment of this vertical between J' and Z'.[60] Suppose she chose J'. Since $J''J$ is less than JZ which is itself less than GJ', a consequence of the fact that MV is steeper than the 45° line, the sum of GJ' and GJ'', is larger than GJ. Our result on the sum of value losses applies, and we contradict the fact that the programme under consideration is a minimum

56 The benchmark plans J_1, J_2 and J_3 correspond to the initial capital stock of unity, but it is clear that these benchmark plans can be constructed for any other initial capital stock, and, loosely speaking, are thereby values of the functions $J_1(J)$, $J_2(J)$ and $J_3(J)$.

57 See the definition of a good programme in footnote 50. Note that the important point is that these losses do not converge to any negative number, however large.

58 The point Z' on $P'M'$ is the analogue of J_1 on PM; namely, it is a one-period plan such that maintenance of full employment next period leads to an unchanged initial stock of capital.

59 We leave it as an exercise for the reader to determine at what point the argument fails.

60 The point Z' on $P'Z'$ is the analogue of J_1 on PM; namely, it is a one-period plan such that maintenance of full employment next period leads to an unchanged initial stock of capital. We have shown through the previous argument that the planner will limit herself to the segment JJ_1 not including J_1; also see footnote 56.

value loss programme. Thus suppose that the planner chooses W'. Then again an initial choice of W rather than Z yields a programme with a lower aggregate minimum value loss. The first step of the argument is complete.

The entire argument can now be repeated for any initial capital stock in the interval (g, g_1^r). Pick the corresponding plan on GG_1^r, mark out the three benchmarks corresponding to it, focus on the segment corresponding to the two benchmarks on the vertical, identify its two sub-segments, and proceed with a consideration of each sub-segment as indicated above. Once the optimal policy function is demonstrated to include the segment GG_1^r, we can show it also to consist of the line $G_1^r D$ by dividing it to intervals corresponding to G, and moving outward, from one interval to the subsequent one. Since we have already shown the desirability of full employment for all stocks in the interval $(0, g)$, the argument is complete.

In conclusion, as in the previous cases, a capital poor economy over-builds and under-depreciates relative to the golden rule stock, but in a way that the discrepancies go to zero in a finite number of time periods, this number depending on the magnitude of the stock of machines that we initially start with. The only role played by the singular stocks g_n^r and g_n^l is that they serve as transition points for the number of periods that it takes for the optimal programme to converge. The dynamic system is "history independent" in terms of Mitra and Nishimura (2001b). The equilibrium is unique and optimal programmes, irrespective of the initial capital stock they begin with, converge to it.[61]

9. Non-uniqueness of optimal policies ($\xi = 1$)

In the case $\xi = 1$ considered in Section 7, we refrained from using the optimal programme to delineate a policy function as consisting of the two lines MV and OD. The reason for this is hinted at in the statement of Theorem 1 in the Appendix: an optimal programme that does not converge to the golden rule stock is not necessarily a unique programme. The point is simply that Brock's (1970) theorem identifies a programme minimizing the aggregate value loss among all programmes starting from a given initial stock as an optimal programme but does not rule out programmes with a positive aggregate value loss as non-optimal programmes. As we shall show in this section, this is dramatically not the case in the one type of machine version of the RSS model.

For concreteness, begin with an initial stock of unity, the abscissa m of the point M in Figure 16. We first show that it is not optimal for the planner to choose any plan on the vertical segment above G'. We proceed in two steps. Under the first, consider a programme which starts at a point anywhere on the interval PM_m, say S_1. If the ordinates of all its subsequent plans are greater than the ordinate of M_g, it can be overtaken by a programme that starts at G' and remains at G for all periods. However, if the ordinate of the second plan of the given programme is greater than the ordinate of M_g, but is below it in some subsequent period, then it can be overtaken by a programme that differs from it only for a finite number of periods: in starting at G', and remaining at G for all those periods at which it is above the horizontal through M_g. If the ordinate of the second plan of the given programme is already below M_g, we are in a special case of the previous possibility, and it can be overtaken by a programme that differs from it only by starting at the point G'.

[61] To be sure, the speed of convergence, and the qualitative feature of the optimal programme (amplitude, monotonic versus non-monotonic convergence etc.), does depend on the initial capital stock, but these features are not given prominence in the definition of "history independent" systems that we are working with.

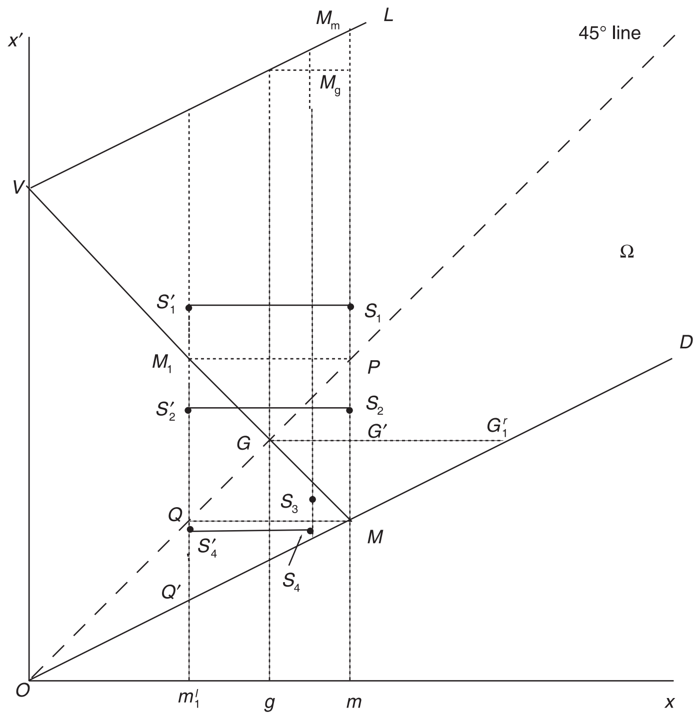

FIGURE 16. Non-uniqueness of optimal programmes $\xi = 1$

The second possibility is that of a programme which starts at a point anywhere on the interval $G'P$ not including G', say S_2. If the ordinate of the second plan of this programme is not less than the ordinate of M, as at S_3, then this programme can be overtaken by another that differs from it only in starting at a plan with the same ordinate as S_3. On the other hand, if this condition does not obtain and the ordinate of the second plan is not greater than the ordinate of M, as at S_4, we need more information about the given programme. If the ordinate of the third plan on the given programme is not less than the ordinate of Q', then it can be overtaken by another programme that differs from it only in the first and second plans: M instead of S_2 and a plan on the vertical through Q and Q' with the same ordinate as S_4. If, on the other hand, the ordinate of the third plan of the given programme is less than the ordinate of Q', and remains less than it in all subsequent periods, then it can be overtaken by a programme that starts at M, and remains at Q. In contrast, if the ordinate of the third plan of the given programme is less than the ordinate of Q', but rises above it in a subsequent period, then it can be overtaken by a programme that differs from it only for a finite number of periods: in starting at M, moving to Q and remaining there for only those periods at which it is below the horizontal through Q'.

Thus any programme starting on the interval $G'M_m$ can be overtaken by a programme starting in the interval MG'. Now consider the initial stock m_1^l and the vertical through Q and Q' that designates it. We shall show in three steps that any programme that does not start at M_1 can be overtaken. First, consider any programme starting on the interval above M_1, say at S_1', and note the plan corresponding to it on MM_m; S_1 in Figure 16. Such a programme can be overtaken by a programme that starts at M_1 and then follows the programme that was constructed to overtake the programme that started at S_1. Second, consider any programme that starts at a plan in the interval M_1Q', not including M_1, say at S_4'. Such a programme can be overtaken by a programme that starts at P' and is identical to it if the ordinate of the second plan of the given programme is above Q, but chooses M and remains at Q for all those periods when this is not so.

The arguments developed for the initial capital stocks m and m_1^l can now be respectively used to apply to any initial capital stock in the intervals (g, m) and (m_1^l, g). This is to say that for any initial stock in (g, m), a programme starting at a plan whose ordinate is above that of G can be overtaken; and asymmetrically, for any initial stock in (m_1^l, g), a programme starting at a plan not on $P'G$ can be overtaken. But now the argument for non-uniqueness of optimal paths is in place. If the programme that starts at G' and remains at G can be overtaken, it must be overtaken by a programme that remains in the triangle $GG'M$. However, this is a contradiction since such a programme has a strictly lower utility sum than the given programme over any two subsequent time periods t and $t + 1$, where t is a positive even integer. But this argument applies to any programme that starts in the interval $G'M$ and by extension, any programme that starts in the triangle $GG'M$.

The argument is complete and we can combine this analysis with that of the previous section to furnish what may perhaps be the most significant finding of this paper; namely, that the optimal policy correspondence for the case $\xi = 1$, is given

$$x(t+1) = \begin{cases} (1/a) - x(t) & \text{when } 0 \le x(t) \le g \\ z \in \{x\colon (1/a) - x(t) \le x \le g\} & \text{when } g \le x(t) \le 1 \\ z \in \{x\colon (1-d)x(t) \le x \le g\} & \text{when } 1 \le x(t) \le g_1^r \\ (1-d)x(t) & \text{when } x(t) \ge g_1^r \end{cases} \tag{21}$$

10. Conclusion and the Nishimura-Yano geometry

We can now offer an algebraic expression of the geometry presented above. On collecting the various cases presented as Equations (17), (18), (20) and (21), we can summarize the optimal policy correspondence as

$$x(t+1) \in \begin{cases} \{x \in \mathbb{R}_+\colon h(x(t)) \le x \le g(x(t))\} & \text{when } \xi = 1 \\ h(x(t)) & \text{when } \xi > 1 \\ g(x(t)) & \text{when } -1 < \xi < 1 \end{cases} \tag{22}$$

where the two real-valued functions h and g on $\mathbb{R}_+$ are given by $h(x) = \max[\xi x + (1/a), \hat{x}, (1-d)x]$ and $g(x) = \max[\xi x + (1/a), (1-d)x]$. As is particularly well understood by theorists of international trade, for consideration of more general models, the geometry facilitates the expression of the solution and furnishes the intuition around which its analytical proofs can be constructed.

This observation leads us to the Leontief two-sector model studied by Nishimura and Yano (1994, 1995, 1996a, 1996b, 2000) and ask how far one could have gone towards obtaining Equation (22) by relying only on their geometrical apparatus. In hindsight, it is also easy to discern leads in the earlier work of Boldrin *et al*. (1961) and Boldrin and Deneckere (1990), Figure 10.5), but it is only in Nishimura and Yano (1996a) that there is full reliance on a linear programming set-up and, in an adoption of a piecewise linear reduced form utility function $u(\cdot,\cdot)$ and explicit avoidance of assumptions on its second derivative, a departure from methods of differential calculus.[62] The authors go beyond illustration to provide a precise depiction of Ω and $u(\cdot,\cdot)$, obtaining in the process an analogue of our *MV* line which they see, and derive, as a full employment, no excess capacity line; see Nishimura and Yano (1995, pp. 986–987) and Nishimura and Yano (1996a, Figures 1 and 7). There is full appreciation of the fact that the qualitative properties of the optimal programme, through their dependence on the slope of *MV*, revolve around the root of the underlying characteristic equation, and that the only complication arises globally from the fact that there is a kink in the difference equation; see Nishimura and Yano (1996a, Figures 1 to 5).[63] However, since the authors assume full depreciation, and that machines are also needed to make machines, their Figure 1 in Nishimura and Yano (1995, 1996a, 2000) is obtained from Figure 1 in this paper by tilting the line *OV* to the right, and making the lines *VL* and *OD* perfectly flat.[64] What is then particularly of interest in the comparison of the two geometries is that the point *V* becomes a "live" kink in their model, constituting the left hand arm of the tent map, and the point *M*, the important kink for us, is ruled out of consideration by a limitation to a restricted range of initial stocks.

This limitation is justified by the fact that Nishimura and Yano are primarily concerned with developing what is now referred to as an anti-turnpike theorem (see Boldrin and Woodford (1990, p. 20)) without differentiability assumptions on the given optimal policy function. As such, the fact that we are concerned with the global characterization of optimal programmes without discounting is a significant difference in objectives and the corresponding geometrical development.[65] In addition to an unlimited range of initial stocks, it shows up in our exploitation of the *MV* line and its parallels as the iso-value loss lines based on golden rule prices. Whereas this aspect plays no role in the Nishimura-Yano geometrical apparatus,[66] we have no use for the subdifferentials of the value and reduced-form utility functions as in Figures 2–4 in Nishimura and Yano (1995, 2000). What is interesting about the problem studied here is that, rather than all trajectories of the dynamic system resulting from the RSS specifications, we have to limit ourselves to those that cannot be overtaken in a well-defined sense. This combination of optimization and difference

[62] The consequences of this are explicitly noted in Nishimura and Yano (1994, p. 994). This being said, we find remarkable the appearance of Figure 10.5 in Boldrin and Deneckere (1990. pp. 248–249) and a reference to how the authors "exploit the interaction between the downward sloping portion of the policy function and the depreciation constraint."

[63] In McKenzie (1987, p. 714), McKenzie writes "The behavior of paths on [the von Neumann facet] *F* may be studied by means of difference equations" and refers to his 1963 paper where this was "done explicitly for the generalized Leontief model." Also, students of dynamic systems are well aware of the difficulties, and consequent wealth of dynamic patterns, arising from functions with kinks; in addition to Milnor and Thurston (1977) and Collet and Eckmann (1980), also see Majumdar *et al.* (2000, Chapter 1).

[64] We leave it to the reader to convince herself that our methods and results all go through for the case of full depreciation.

[65] These two different objectives are noted in Boldrin and Deneckere (1990, p. 229): "[W]e do not construct "artificial" economies that exhibit a pre-chosen dynamics in equilibrium. Rather, we start with a specification of technology and preferences and derive the implied dynamics."

[66] However, it worthy of emphasis that the value loss method plays a crucial role in the Nishimura-Yano geometry (see Nishimura and Yano (1996a, Lemma 2; 2000, Lemma 4.2)); only that in line with their objectives, they do not work only with golden rule prices.

equations in a linear framework, iso-value lines combined with cobweb-like diagrams, leads to a situation that is double edged. It makes things easier, as in Section 8 where there is particularly fast convergence despite, or perhaps because of, a root greater than unity; and also more difficult, as in Section 7, where we have a uncountable multiplicity of optimum paths, giving the planner substantial "room to manoeuvre". Whereas the first is not identified by Nishimura and Yano (1995, pp. 264–265), it is interesting that the second non-uniqueness problem is explicitly recognized even in a setting with discounting.[67] Thus, our comparison of the two geometries, Euclidean versus Cartesian, naturally leads to two questions: first, to provide a (global) characterization of optimal programmes in a two-sector model with Leontief production functions in an undiscounted case; and second, in a converse move now clearly seen as pertaining only to the discounted setting, to extend the validity of the anti-turnpike theorem to the unimodal, but only partly expansive, functions delineated here. The first problem is one where "cones of diversification" are given fuller play, and is tackled in Fujio (2004); the second is left open for future work.[68]

Appendix

The theorem presented below relies on results in Khan and Mitra (2003) on the existence of an optimal programme in the RSS model. These results apply the methods of Brock (1970). The reader should note from the discussion in Section 4 that the standing hypotheses in (Khan and Mitra (2003), Condition (1)) is automatically fulfilled in the one type machine case that we are considering here.

Theorem 1: *For any arbitrary initial stock,* $x_0 \geq 0$*, there exists an optimal programme from* x_0*. If the initial stock* x_0 *equals the golden rule stock* $\hat{x} = \hat{y} = 1/(1 + da)$*, then the stationary programme* $\{\hat{x}, \hat{y}\}$ *is an optimal programme from* x_0*. Finally, if* $\xi \neq 1$ *in the interval* $(-1, \infty)$*, a programme that uniquely minimizes the aggregate value loss of all programmes starting from* x_0 *is the unique optimal programme.*

Before a proof of the theorem, we shall need the following lemma.

Lemma 1: *For all* $\xi \neq 1$ *in the interval* $(-1, \infty)$*, every good programme converges to the golden rule stock.*

Proof: From Proposition 8 in Khan and Mitra (2003) we know that any good programme converges to the von Neumann facet, and from the proof of Proposition 11 in Khan and Mitra (2003) that it converges to the golden rule stock in the case $-1 < \xi < 1$. For the case, $\xi > 1$, we leave it to the reader to construct a proof based on Figure 15a.

We can now provide a proof.

[67] Nishimura and Yano write, "Because we deal with Leontief production functions, it might appear easier to handle the present model. . . . However, [this] creates an extra difficulty."

[68] Subsequent to this paper, in Khan and Mitra (2005), the authors provide algebraic validation of these results through alternative mathematical techniques.

Proof of Theorem 1: The first two claims follow from Theorem 2 in Khan and Mitra (2003). We turn to a proof of the third claim. Let the programme $\{x(t), y(t)\}$ with $x(0) = x_0$ be such that its associated value loss sequence $\{\delta(t)\}$ has the unique minimum aggregate among all programmes starting from x_0. Since Proposition 2 in Khan and Mitra (2003) guarantees that there exists a good programme from any initial stock x_0, certainly $\{x(t), y(t)\}$ is a good programme.

Let $\{x'(t), y'(t)\}$ be another programme starting from x_0 that is optimal and has an associated value loss sequence $\{\delta'(t)\}$. From Proposition 10 in Khan and Mitra (2003), we know that $\{x'(t), y'(t)\}$ is a good programme.

We now appeal to Proposition 6 in Khan and Mitra (2003) to assert that for all positive integers T,

$$\sum_{t=0}^{T}(by(t) - by'(t)) = \hat{p}(x'(T+1) - x(T+1)) + \sum_{t=0}^{T}\delta'(t) - \sum_{t=0}^{T}\delta(t).$$

By hypothesis, there exists $2\varepsilon > 0$ and a positive integer T_1 such that for all $T > T_1$, $\sum_{t=0}^{T}\delta'(t) - \sum_{t=0}^{T}\delta(t) > 2\varepsilon$. Furthermore, since $\{x(t), y(t)\}$ and $\{x'(t), y'(t)\}$ are both good programmes, we can appeal to Lemma 1 to assert the existence of a positive integer T_2 such that for all $T > T_2$, $\hat{p}(x'(T+1) - x(T+1)) > -\varepsilon$. Putting these assertions together, we can conclude that

$$\sum_{t=0}^{T}(by(t) - by'(t)) > \varepsilon \qquad \text{for all } T > \max\{T_1, T_2\},$$

a contradiction to the optimality of the programme $\{x'(t), y'(t)\}$. The proof is complete.

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